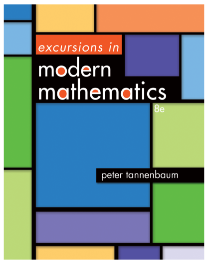

Chapter 2: The Mathematics of Power

2.1 An Introduction to Weighted Voting

Weighted Voting

In a democracy we take many things for granted, not the least of which is the idea that we are all equal. When it comes to voting rights, the democratic ideal of equality translates into the principle of one person-one vote. But is the principle of one person-one vote always fair? Should one person-one vote apply when the voters are institutions or governments, rather than individuals?

Weighted Voting

What we are talking about here is the exact opposite of the principle of one voter–one vote, a principle best described as one voter–x votes and formally called weighted voting. Weighted voting is not uncommon; we see examples of weighted voting in shareholder votes in corporations, in business partnerships, in legislatures, in the United Nations, and, most infamously, in the way we elect the President of the United States.

Electoral College

The Electoral College consists of 51 "voters" (each of the 50 states plus the District of Columbia), each with a weight determined by the size of its Congressional delegation (number of Representatives and Senators). At one end of the spectrum is heavyweight California (with 55 electoral votes); at the other end of the spectrum are lightweights like Wyoming, Montana, North Dakota, and the District of Columbia (with a paltry 3 electoral votes). The other states fall somewhere in between.

Electoral College

The 2000 and 2004 presidential elections brought to the surface, in a very dramatic way, the vagaries and complexities of the Electoral College system, and in particular the pivotal role that a single state (Florida in 2000, Ohio in 2004) can have in the final outcome of a presidential election.

In this chapter we will look at the mathematics behind weighted voting, with a particular focus on the question of power.

Weighted Voting System

We will use the term **weighted voting system** to describe any formal voting arrangement in which voters are not necessarily equal in terms of the number of votes they control. We will only consider voting on yes–no votes, known as motions. Note that any vote between two choices (say A or B) can be rephrased as a yes–no vote (a Yes is a vote for A, a No is a vote for B). Every weighted voting system is characterized by three elements:

Weighted Voting System

The players.

We will refer to the voters in a weighted voting system as players. Note that in a weighted voting system the players may be individuals, but they may also be corporations, governmental agencies, states, municipalities, or countries. We will use N to denote the number of players in a weighted voting system, and typically (unless there is a good reason not to) we will call the players $P_1, P_2, \ldots, P_N$.

 PEARSON 2.1-7

Weighted Voting System

The weights.

The hallmark of a weighted voting system is that each player controls a certain number of votes, called the weight of the player. We will assume that the weights are all positive integers, and we will use $w_1, w_2, \ldots, w_N$ to denote the weights of $P_1, P_2, \ldots, P_N$, respectively. We will use $V = w_1 + w_2 + \ldots + w_N$ to denote the total number of votes in the system.

 PEARSON 2.1-8

Weighted Voting System

The quota.

In addition to the player's weights, every weighted voting system has a **quota**. The quota is the *minimum number of votes required to pass a motion*, and is denoted by the letter q. While the most common standard for the quota is a *simple majority* of the votes, the quota may very well be something else.

 PEARSON 2.1-9

Weighted Voting System

In the U.S. Senate, for example, it takes a simple majority to pass an ordinary law, but it takes a minimum of 60 votes to stop a filibuster, and it takes a minimum of two-thirds of the votes to override a presidential veto.

In other weighted voting systems the rules may stipulate a quota of three-fourths of the votes, or four-fifths, or even unanimity (100% of the votes).

 PEARSON 2.1-10

Notation

The standard notation used to describe a weighted voting system is to use square brackets and inside the square brackets to write the quota q first (followed by a colon) and then the respective weights of the individual players separated by commas. It is convenient and customary to list the weights in numerical order, starting with the highest, and we will adhere to this convention throughout the chapter.

 PEARSON 2.1-11

Notation

Thus, a generic weighted voting system with N players can be written as:

GENERIC WEIGHTED VOTING SYSTEM WITH *N* PLAYERS

$[q: w_1, w_2, \ldots, w_N]$

(with $w_1 \geq w_2 \geq \ldots \geq w_N$)

 PEARSON 2.1-12

Example 2.1 Venture Capitalism

Four partners (P_1, P_2, P_3, and P_4) decide to start a new business venture. In order to raise the $200,000 venture capital needed for startup money, they issue 20 shares worth $10,000 each. Suppose that P_1 buys 8 shares, P_2 buys 7 shares, P_3 buys 3 shares, and P_4 buys 2 shares, with the usual agreement that one share equals one vote in the partnership. Suppose that the quota is set to be two-thirds of the total number of votes.

Example 2.1 Venture Capitalism

Since the total number of votes in the partnership is V = 20, and two-thirds of 20 is 13 1/3, we set the quota to q = 14. Using the weighted voting system notation we just introduced, the partnership can be described mathematically as [14: 8, 7, 3, 2].

Example 2.2 Anarchy

Imagine the same partnership discussed in Example 2.1, with the only difference being that the quota is changed to 10 votes. We might be tempted to think of this partnership as the weighted voting system [10:8,7,3,2], but there is a problem here: the quota is too small, making it possible for both Yes's and No's to have enough votes to carry a particular motion.

Example 2.2 Anarchy

(Imagine, for example, that an important decision needs to be made and P_1 and P_4 vote yes and P_2 and P_3 vote no. Now we have a stalemate, since both the Yes's and the No's have enough votes to meet the quota.)

In general when the quota requirement is less than simple majority ($q \leq V/2$) we have the potential for both sides of an issue to win–a mathematical version of anarchy.

Example 2.3 Gridlock

Once again, let's look at the partnership introduced in Example 2.1, but suppose now that the quota is set to q = 21, more than the total number of votes in the system. This would not make much sense. Under these conditions no motion would ever pass and nothing could ever get done–a mathematical version of gridlock.

The Range of Values of the Quota

Given that we expect our weighted voting systems to operate without anarchy or gridlock, from here on we will assume that *the quota will always fall somewhere between simple majority and unanimity of votes*. Symbolically, we can express these requirements by two inequalities: $q > V/2$ and $q \leq V$. These two restrictions define the range of possible values of the quota:

The Range of Values of the Quota

RANGE OF VALUES OF THE QUOTA

$V/2 < q \leq V$

(where $V = w_1 + w_2 + \ldots + w_N$)

Example 2.4 One Partner–One Vote?

Let's consider the partnership introduced in Example 2.1 one final time. This time the quota is set to be $q = 19$. Here we can describe the partnership as the weighted voting system [19: 8, 7, 3, 2]. What's interesting about this weighted voting system is that the *only way a motion can pass is by the unanimous support of all the players*. (Note that P_1, P_2, and P_3 together have 18 votes–they still need P_4's votes to pass a motion.)

Example 2.4 One Partner–One Vote?

In a practical sense this weighted voting system is no different from a weighted voting system in which each partner has 1 vote and it takes the unanimous agreement of the four partners to pass a motion (i.e., [4: 1, 1, 1, 1]).

Equal Say versus Equal Number of Votes

The surprising conclusion of Example 2.4 is that the weighted voting system [19: 8, 7, 3, 2] describes a one person–one vote situation in disguise. This seems like a contradiction only if we think of one person–one vote as implying that all players have an equal number of votes rather than an equal say in the outcome of the election. Apparently, these two things are not the same! As Example 2.4 makes abundantly clear, just looking at the number of votes a player owns can be very deceptive.

Example 2.5 Dictators

Consider the weighted voting system [11: 12, 5, 4]. Here one of the players owns enough votes to carry a motion single handedly. In this situation P_1 is in complete control–if P_1 is for the motion, then the motion will pass; if P_1 is against it, then the motion will fail. Clearly, in terms of the power to influence decisions, P_1 has *all* of it. Not surprisingly, we will say that P_1 is a *dictator*.

Dictator and Dummies

In general a player is a **dictator** if *the player's weight is bigger than or equal to the quota*. Since the player with the highest weight is P_1, we can conclude that *if there is a dictator, then it must be* P_1. When P_1 is a dictator, all the other players, regardless of their weights, have absolutely no say on the outcome of the voting–there is never a time when their votes really count. A player who never has a say in the outcome of the voting is a player who has no power, and we will call such players **dummies**.

Example 2.6 Unsuspecting Dummies

Four college friends (P_1, P_2, P_3, and P_4) decide to go into business together. Three of the four (P_1, P_2, and P_3) invest $10,000 each, and each gets 10 shares in the partnership. The fourth partner (P_4) is a little short on cash, so he invests only $9000 and gets 9 shares. As usual, one share equals one vote.
The quota is set at 75%, which here means q = 30 out of a total of V = 39 votes. Mathematically (i.e., stripped of all the irrelevant details of the story), this partnership is just the weighted voting system [30: 10, 10, 10, 9].

Example 2.6 Unsuspecting Dummies

Everything seems fine with the partnership until one day P_4 wakes up to the realization that with the quota set at q = 30 he is completely out of the decision-making loop: For a motion to pass P_1, P_2, and P_3 all must vote Yes, and at that point it makes no difference how P_4 votes. Thus, there is never going to be a time when P_4's votes are going to make a difference in the final outcome of a vote. Surprisingly, P_4–with almost as many votes as the other partners–is just a dummy!

Example 2.7 Veto Power

Consider the weighted voting system [12: 9, 5, 4, 2]. Here P_1 plays the role of a "spoiler"–while not having enough votes to be a dictator, the player has enough votes to *prevent a motion from passing*. This happens because if we remove P_1's 9 votes the sum of the remaining votes (5 + 4 + 2 = 11) is less than the quota q = 12. Thus, even if all the other players voted Yes, without P_1 the motion would not pass. In a situation like this we say that has *veto power*.

Veto Power

A player who is not a dictator has **veto power** if a *motion cannot pass unless the player votes in favor of the motion*. In other words, a player with veto power cannot force a motion to pass (the player is not a dictator), but can force a motion to *fail*. If we let w denote the weight of a player with veto power, then the two conditions can be expressed mathematically by the inequalities $w < q$ (the player is not a dictator), and $V - w < q$ (the remaining votes in the system are not enough to pass a motion).

Veto Power

VETO POWER

A player with weight w has veto power if and only if $w < q$ and $V - w < q$ (where $V = w_1 + w_2 + \ldots + w_N$).